THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2050C Mathematical Analysis I

Tutorial 7 (March 18)

Definition (Contractive Sequences)**.** We say that a sequence (x_n) of real numbers is **contractive** if there exists a constant C, $0 < C < 1$, such that

$$|x_{n+2} - x_{n+1}| \leq C|x_{n+1} - x_n| \qquad \text{for all } n \in \mathbb{N}. \tag{\#}$$

The number C is called the **constant** of the contractive sequence.

Remark. Do not confuse (#) with the following condition:

$$|x_{n+2} - x_{n+1}| < |x_{n+1} - x_n| \qquad \text{for all } n \in \mathbb{N}. \tag{\#\#}$$

For example, $(\sqrt{n})$ satisfies (##) but it is not contractive.

Theorem. *Every contractive sequence is a Cauchy sequence, and therefore is convergent.*

Example 1. (Sequence of Fibonacci Fractions) Consider the Fibonacci fractions $x_n := f_n/f_{n+1}$, where (f_n) is the Fibonacci sequence defined by $f_1 = f_2 = 1$ and $f_{n+2} := f_{n+1} + f_n$, $n \in \mathbb{N}$. Show that the sequence (x_n) converges to $1/\varphi$, where $\varphi := (1 + \sqrt{5})/2$ is the Golden Ratio.

Example 2. Show that if (x_n) is unbounded, then there exists a subsequence (x_{n_k}) such that $\lim(1/x_{n_k}) = 0$.

Solution. As (x_n) is unbounded, we have $\forall M > 0$, $\exists\, n \in \mathbb{N}$ such that $|x_n| > M$.

Pick $n_1 \in \mathbb{N}$ such that $|x_{n_1}| > 1$.
Then pick $n_2 \in \mathbb{N}$ such that $|x_{n_2}| > \max\{2, |x_1|, |x_2|, \ldots, |x_{n_1}|\}$. So $|1/x_{n_2}| < 1/2$ and $n_2 > n_1$.
Suppose $n_1 < n_2 < \cdots < n_k$ are chosen. Pick $n_{k+1} \in \mathbb{N}$ such that $|x_{n_{k+1}}| > \max\{k + 1, |x_1|, |x_2|, \ldots, |x_{n_k}|\}$. So $|1/x_{n_{k+1}}| < 1/(k + 1)$ and $n_{k+1} > n_k$.
Continue in this way, we obtain a subsequence (x_{n_k}) of (x_n) such that

$$|1/x_{n_k}| < 1/k \qquad \text{for all } k \in \mathbb{N}.$$

Now $\lim(1/x_{n_k}) = 0$ follows immediately from Squeeze Theorem.

◀

Classwork

1. Let $x_n := \sqrt{n}$. Show that (x_n) satisfies $\lim |x_{n+1} - x_n| = 0$, but that it is not a Cauchy sequence by definition.

   Solution. Clearly, $\lim |x_{n+1} - x_n| = \lim |\sqrt{n+1} - \sqrt{n}| = \lim \left(\frac{1}{\sqrt{n+1}+\sqrt{n}}\right) = 0$.

   Take $\varepsilon_0 = \sqrt{2} - 1 > 0$. Then for any $n \in \mathbb{N}$,
   $$|x_{2n} - x_n| = |\sqrt{2n} - \sqrt{n}| = (\sqrt{2} - 1)\sqrt{n} \geq \varepsilon_0.$$
   Hence (x_n) is not Cauchy. ◀

2. Let (x_n) be a sequence of real numbers defined by
   $$\begin{cases} x_1 = 1, \quad x_2 = 2, \\ x_{n+2} := \dfrac{1}{3}(2x_{n+1} + x_n) \quad \text{for all } n \in \mathbb{N}. \end{cases}$$
   Show that (x_n) is convergent and find its limit.

   Solution. Note that
   $$x_{n+2} - x_{n+1} = \frac{1}{3}(2x_{n+1} + x_n) - x_{n+1} = -\frac{1}{3}(x_{n+1} - x_n).$$
   In particular,
   $$|x_{n+2} - x_{n+1}| = \frac{1}{3}|x_{n+1} - x_n| \qquad \text{for } n \in \mathbb{N},$$
   so (x_n) is contractive, hence convergent. As
   $$x_{n+2} - x_{n+1} = -\frac{1}{3}(x_{n+1} - x_n) = \cdots = (-\frac{1}{3})^n(x_2 - x_1) = (-\frac{1}{3})^n.$$
   we have
   $$\sum_{k=0}^{n}(x_{k+2} - x_{k+1}) = \sum_{k=0}^{n}(-\frac{1}{3})^k$$
   $$x_{n+2} - x_1 = \frac{1 - (-\frac{1}{3})^{n+1}}{1 - (-\frac{1}{3})}.$$
   Hence $\lim(x_n) = \lim\left(1 + \frac{3}{4}(1 - (-\frac{1}{3})^{n+1})\right) = \frac{7}{4}$. ◀